"13º μάθημα" 10/01/22

Ασκ 4 α.γ.π : $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$, με $i = 1, \dots, n$

(i) $r_{xy}^2 = R^2$, (ii) $r_{Y,\hat{Y}}^2 = r_{X,Y}^2$

Λύση :

(i) $$R^2 = \frac{SSreg}{SStot} = \frac{\Sigma(\hat{Y}_i - \bar{Y})^2}{\Sigma(Y_i - \bar{Y})^2} = \frac{\Sigma(\hat{\beta}_0 + \hat{\beta}_1 X_i - \cancel{\bar{Y}})^2}{\Sigma(Y_i - \bar{Y})^2} =$$

$$= \frac{\hat{\beta}_1^2 \cdot \Sigma(X_i - \bar{X})^2}{\Sigma(Y_i - \bar{Y})^2} \quad (1)$$

$$\hat{\beta}_1 = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\Sigma(X_i - \bar{X})^2} , \quad \boxed{r_{X,Y} = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\Sigma(X_i - \bar{X})^2}\sqrt{\Sigma(Y_i - \bar{Y})^2}}}$$

$$\Rightarrow \hat{\beta}_1 = \left[\frac{\Sigma(Y_i - \bar{Y})^2}{\Sigma(X_i - \bar{X})^2}\right]^{1/2} \cdot r_{X,Y} \quad (2)$$

Από (1) και (2) : $$R^2 = \frac{\cancel{\Sigma(Y_i - \bar{Y})^2}}{\cancel{\Sigma(X_i - \bar{X})^2}} \cdot r_{X,Y}^2 \cdot \frac{\cancel{\Sigma(X_i - \bar{X})^2}}{\cancel{\Sigma(Y_i - \bar{Y})^2}} = r_{X,Y}^2$$

(ii) $$r_{Y,\hat{Y}}^2 = \frac{\left[\Sigma(Y_i - \bar{Y}) \cdot (\hat{Y}_i - \bar{\hat{Y}}_i)\right]^2}{\Sigma(Y_i - \bar{Y})^2 \cdot \Sigma(\hat{Y}_i - \bar{\hat{Y}}_i)^2} =$$ Πως εκφράζεται αυτή η ποσότητα?

$$\bar{\hat{Y}}_i = \frac{1}{n} \Sigma \hat{Y}_i = \frac{1}{n} \Sigma(\hat{\beta}_0 + \hat{\beta}_1 X_i) = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}$$

$$= \frac{\left[\Sigma(Y_i - \bar{Y})(\hat{Y}_i - \hat{\beta}_0 - \hat{\beta}_1 \bar{X})\right]^2}{\Sigma(Y_i - \bar{Y})^2 \cdot \Sigma(\hat{Y}_i - \hat{\beta}_0 - \hat{\beta}_1 \bar{X})^2} = \frac{\left[\Sigma(Y_i - \bar{Y})(\cancel{\hat{\beta}_0} + \hat{\beta}_1 X_i - \cancel{\hat{\beta}_0} - \hat{\beta}_1 \bar{X})\right]^2}{\Sigma(X_i - \bar{X})^2 \cdot \Sigma(\hat{\beta}_0 + \hat{\beta}_1 X_i - \hat{\beta}_0 - \hat{\beta}_1 \bar{X})^2}$$

$$= \frac{\cancel{\hat{\beta}_1^2} \cdot \left[\Sigma(Y_i-\bar{Y}).(X_i-\bar{X})\right]^2}{\Sigma(X_i-\bar{X})^2 \cdot \cancel{\hat{\beta}_1^2} \cdot \Sigma(X_i-\bar{X})^2} = \frac{\left[\Sigma(Y_i-\bar{Y})(Y_i-\bar{X})\right]^2}{\Sigma(X_i-\bar{X})^2 \cdot \Sigma(Y_i-\bar{Y})^2} = r_{X,Y}^2$$

Ασκ. 7 Έστω Y_1, Y_2 ανεξ τμ με:

$E(Y_1)=\theta$, $E(Y_2)=2\theta$, με θ πραγμ. παράμετρο

Χρησιμοποιώντας α.γ.π, να βρεθεί:

(i) ΕΕΤ της θ

(ii) SSres

Λύση:

i) $\left.\begin{array}{l} E(Y_1)=\theta \\ E(Y_2)=2\theta \end{array}\right\} \Rightarrow E\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} EY_1 \\ EY_2 \end{pmatrix} = \begin{pmatrix} \theta \\ 2\theta \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\theta$

Θεωρώ το μοντέλο α.γ.π:

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\theta + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \quad \text{με} \quad E\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Έστω $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$, όπου $\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\underset{\sim}{\beta} = \theta$ και $\underset{\sim}{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$

Γνωρίζω ότι οι ΕΕΤ $\hat{\underset{\sim}{\beta}} = (X'X)^{-1} \cdot X' \cdot \underset{\sim}{Y}$

$$SSres = \underset{\sim}{Y}'\underset{\sim}{Y} - \hat{\underset{\sim}{\beta}}' \cdot X' \cdot \underset{\sim}{Y}$$

$$\hat{\theta} = \left((1,2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)^{-1} \cdot (1,2)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} =$$

$$= (5)^{-1} \cdot (1Y_1 + 2Y_2) \Rightarrow \boxed{\hat{\theta} = \frac{1}{5}(Y_1 + 2Y_2)}$$

ii) $SS_{res} = (y_1, y_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \frac{1}{5}(y_1+2y_2)(1,2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} =$

$= y_1^2 + y_2^2 - \frac{1}{5}(y_1+2y_2)(y_1+2y_2) =$

$= y_1^2 + y_2^2 - \frac{1}{5}(y_1+2y_2)^2.$

Ασκ. 9

$\underset{\sim}{y} = X \cdot \underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ με $X = \begin{pmatrix} 1 & x & \cdots & x \\ 1 & x & \cdots & x \\ \vdots & \vdots & & \\ 1 & x & \cdots & x \end{pmatrix}$

$\Sigma e_i = 0$ ή $\Sigma(y_i - \hat{y}_i) = 0$

σωστηρίες αφού δν έχω κάτι άλλο

Χρήση κ.ε $(X'X) \cdot \underset{\sim}{\beta} = X' \cdot \underset{\sim}{y}$ οποίε επαληθεύεται από τη λύση τους που είναι $\hat{\underset{\sim}{\beta}}$ και έτσι:

$(X'X)\hat{\underset{\sim}{\beta}} = X' \cdot \underset{\sim}{y} \Rightarrow X' \cdot \underset{\sim}{y} - (X'X) \cdot \hat{\underset{\sim}{\beta}} = \underset{\sim}{0} \Rightarrow$

$\Rightarrow X'(\underset{\sim}{y} - X \cdot \hat{\underset{\sim}{\beta}}) = \underset{\sim}{0} \Rightarrow X'(\underset{\sim}{y} - \hat{\underset{\sim}{y}}) = \underset{\sim}{0} \Rightarrow$

$(\underset{\sim}{y} - \hat{\underset{\sim}{y}})' \cdot X = \underset{\sim}{0} \Rightarrow$

Όμως $\underset{\sim}{y} - \hat{\underset{\sim}{y}} = (y_1 - \hat{y}_1, y_2 - \hat{y}_2, \ldots, y_n - \hat{y}_n)$

και $X = \begin{pmatrix} 1 & x & x & \cdots & x \\ 1 & x & x & \cdots & x \\ \vdots & \vdots & \vdots & & \\ 1 & x & x & \cdots & x \end{pmatrix}$

Άρα: $\underbrace{(y_1 - \hat{y}_1, y_2 - \hat{y}_2, \ldots, y_n - \hat{y}_n)}\begin{pmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\Rightarrow \sum_{i=1}^{n} (y_i - \hat{y}_i) \cdot 1 = 0 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n} (y_i - \hat{y}_i) = 0 \Rightarrow \sum (y_i - \hat{y}_i) = 0 \Rightarrow \sum e_i = 0$$

Ασκ 10

Π.γ.Π $\underset{\sim}{y} = X \cdot \underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$, $\text{rank}(X) = p+1$

Νδο: α) $\sum_{i=1}^{n} \hat{y}_i (y_i - \hat{y}_i) = 0$

β) $\sum_{i=1}^{n} Var(\hat{y}_i) = \sigma^2 (p+1)$

<u>Λύση</u>:

α) Από Ασκ 9 $(\underset{\sim}{y} - \underset{\sim}{\hat{y}})' \cdot X = \underset{\sim}{0} \Rightarrow (\underset{\sim}{y} - \underset{\sim}{\hat{y}})' \cdot X \cdot \underset{\sim}{\hat{\beta}} = \underset{\sim}{0} \cdot \underset{\sim}{\hat{\beta}} \Rightarrow$

$$\Rightarrow (\underset{\sim}{y} - \underset{\sim}{\hat{y}})' \cdot \underset{\sim}{\hat{y}} = 0 \Rightarrow \sum_{i=1}^{n} (y_i - \hat{y}_i) \hat{y}_i = 0$$

β) $\underset{\sim}{\hat{y}_i} = \begin{pmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_n \end{pmatrix}$, $Var(\underset{\sim}{\hat{y}}) = \begin{pmatrix} Var(\hat{y}_1) & & Cov(\hat{y}_i, \hat{y}_j) \\ \vdots & Var(\hat{y}_2) & \\ Cov(\hat{y}_j, \hat{y}_i) & & \ddots \; Var(\hat{y}_n) \end{pmatrix}$

πίνακας συνδιακυμάνσεων → πάντα συμμετρικός.

trace = ίχνος

$$\sum_{i=1}^{n} Var(\hat{y}_i) = tr(Var(\underset{\sim}{\hat{y}_i})) =$$

$$\boxed{\text{Γνωρίζουμε ότι: } \underset{\sim}{\hat{y}} = X \cdot \underset{\sim}{\hat{\beta}} \rightarrow Var(\underset{\sim}{y}) = X [\sigma^2 (X'X)^{-1} \cdot X'] = \sigma^2 [X(X'X)^{-1} X']}$$

$$= tr[\sigma^2 X (X'X)^{-1} \cdot X'] = \sigma^2 \cdot tr[\underbrace{X}_{A} \underbrace{(X'X)^{-1} \cdot X'}_{B'}] =$$

$$\xlongequal[tr(AB') = tr(A'B)]{tr(A'B) = tr(AB')} \sigma^2 \cdot tr[\underset{(p+1)\times n}{X'} \cdot \underset{n\times(p+1)}{X} (X'X)^{-1}] = \sigma^2 \cdot tr[I_{(p+1)}]$$

$$= \sigma^2 \cdot \sum_{i=1}^{p+1} 1 = \sigma^2 (p+1)$$

Ασκ 11

$y_i = b_0 + b_1 X_{i1} + b_2 X_{i2} + \varepsilon_i$ με $i = 1, \dots, n$ με $\varepsilon_i \sim N(0, \sigma^2)$

Οι κ.ε. είναι:

$$\boxed{\begin{aligned} 10\hat{b}_0 + 2\hat{b}_1 - 6\hat{b}_2 &= 4 \\ 2\hat{b}_0 + 2\hat{b}_1 \quad\quad &= 6 \\ -6\hat{b}_0 + \quad\quad + 5\hat{b}_2 &= 7 \end{aligned}}$$

$\hat{b} = (x'x)^{-1} \cdot x' \cdot y$

α) Αν $n = 10$, $\Sigma y_i^2 = 107$, να βρεθεί $\hat{b}$, MSres

β) F-τεστ $H_0: b_1 = 2b_2$

γ) Κατάσταση t-τεστ $H_0: b_1 = 2b_2$

Λύση: Γνωρίζω ότι κ.ε: $\boxed{(x'x)\hat{b} = x' \cdot y}$

α) Άρα συμφωνα από τα δεδομένα έχω:

$$x'x = \begin{pmatrix} 10 & 2 & 6 \\ 2 & 2 & 0 \\ -6 & 0 & 5 \end{pmatrix}$$

$$(x' \cdot x)^{-1} = \frac{1}{8} \cdot \begin{pmatrix} 10 & -10 & 12 \\ -10 & 14 & -12 \\ 12 & -12 & 16 \end{pmatrix}$$

$$\underset{(p+1)\times n}{x'} \cdot \underset{n \times 1}{y} = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}$$

Άρα: $\hat{b} = (x' \cdot x)^{-1} \cdot x' \cdot y = \left(\cdots \right) \cdot \left(\cdots \right) = \begin{pmatrix} 8 \\ -5 \\ 11 \end{pmatrix}$

$$MS_{res} = \frac{1}{n-p-1} \cdot SS_{res} = \frac{1}{10-2-1} \cdot \Big(\underbrace{\underset{\sim}{y}' \underset{\sim}{y}}_{\sum y_i^2} - \underset{\sim}{\hat{b}}' x' \underset{\sim}{y} \Big) = 4$$

(β) Γνωρίζω τη γενική γραμμική υπόθεση

$H_0: A \cdot \underset{\sim}{b} = \underset{\sim}{c}$ όπου A είναι $q \times (p+1)$
και $\underset{\sim}{c}$ $q \times 1$

Για έλεγχο της $H_0: A \cdot \underset{\sim}{b} = \underset{\sim}{c}$ η ΣΣΤ είναι:

$$F = \frac{n-p-1}{q} \cdot \frac{(A \cdot \underset{\sim}{\hat{b}} - \underset{\sim}{c})' (A (x' x)^{-1} A') (A \cdot \underset{\sim}{\hat{b}} - \underset{\sim}{c})}{SS_{res}} \sim F_{q, n-p-1}$$

και Κ.Π. $F \geq F_{q, n-p-1, \alpha}$

$H_0: b_1 = 2b_2 \Leftrightarrow H_0: A \cdot \underset{\sim}{b} = \underset{\sim}{c}$ για $\underset{1 \times (p+1)}{A} = (0, 1, -2)$
$\Updownarrow$
$H_0: 0b_0 + 1b_1 - 2b_2 = 0$
με $p = 2$

και $\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Άρα $n = 10$ $p = 2$ $A = (0, 1, -2)$ και $\underset{\sim}{c} = (0, 0, 0)$

(γ) $H_0: b_1 = 2b_2 \Leftrightarrow H_0: \underset{\sim}{a}' \cdot \underset{\sim}{b} = \underset{\sim}{c}$ όπου $a = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Κατασκευάζω t-τεστ

Θα στηριχτώ σε εκτιμητή της $\underset{\sim}{a}' \cdot \underset{\sim}{b}$ δηλ. $\underset{\sim}{a}' \underset{\sim}{\hat{b}}$

Αλλά $\underset{\sim}{\hat{b}} = \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} \sim N_3\left(\underset{\sim}{b} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}, \sigma^2 (x' x)^{-1} \right)$

Άρα: $\underset{\sim}{\alpha}' \cdot \hat{\underset{\sim}{b}} \sim N\left(\underset{\sim}{\alpha}' \underset{\sim}{b} \,,\, \sigma^2 \cdot \underset{\sim}{\alpha}'(X'X)^{-1} \underset{\sim}{\alpha} \right)$

$$\Downarrow$$

$$\frac{\underset{\sim}{\alpha}' \cdot \hat{\underset{\sim}{b}} - \underset{\sim}{\alpha}' \cdot \underset{\sim}{b}}{\sigma \sqrt{\underset{\sim}{\alpha}'(X'X)^{-1}\underset{\sim}{\alpha}}} \sim N(0,1)$$

και υπό την Ho: $\underset{\sim}{\alpha}' \cdot \underset{\sim}{b} = 0$, το $\dfrac{\underset{\sim}{\alpha}' \hat{\underset{\sim}{b}}}{\sigma \sqrt{\underset{\sim}{\alpha}'(X'X)\alpha}} \sim N(0,1)$

υπό την Ho.

Και επειδή $\underset{\sim}{\alpha}'(X'X) \cdot \underset{\sim}{\alpha} = 15{,}75$, έχω:

$$\frac{\underset{\sim}{\alpha}' \cdot \hat{\underset{\sim}{b}}}{\sigma\sqrt{15{,}75}} \sim N(0,1) \text{ υπό την Ho.}$$

Αλλά $\dfrac{SSres}{\sigma^2} \sim X^2_{n-p-1}$ και $\dfrac{SSres}{\sigma^2}$ ανεξ. του $\hat{\underset{\sim}{b}}$.

$$\frac{\dfrac{\underset{\sim}{\alpha}' \cdot \hat{\underset{\sim}{b}}}{\sigma\sqrt{15{,}75}}}{\sqrt{\dfrac{SSres}{\sigma^2}/(n-p-1)}} \sim \frac{N(0,1)}{\sqrt{X^2_{n-p-1}/(n-p-1)}} \sim t_{n-p-1}$$

$$\parallel$$

$$\frac{\underset{\sim}{\alpha}' \cdot \hat{\underset{\sim}{b}}}{\sqrt{15{,}75 \cdot MSres}} \sim t_{n-p-1} \Rightarrow \frac{\hat{b}_1 - 2\hat{b}_2}{\sqrt{15{,}75 \cdot MSres}} \sim t_{n-p-1} \text{ υπό Ho}$$

ΣΣΤ

κ.π. $|t| \geq t_{n-p-1, \frac{\alpha}{2}}$

Ασκ. 21

Έστω οι παρατηρήσεις: $y_1 = \alpha_1 + \varepsilon_1$

$y_2 = 2\alpha_1 - \alpha_2 + \varepsilon_2$

$y_3 = \alpha_1 + 2\alpha_2 + \varepsilon_3$

με $\underset{\sim}{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} \sim N(\underset{\sim}{0}, \sigma^2 \cdot I_3)$

$H_0: \alpha_1 = \alpha_2$

$\underset{\sim}{y} = x \cdot \underset{\sim}{b} + \underset{\sim}{\varepsilon}$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

και $H_0: \alpha_1 = \alpha_2$ ισοδύναμη με $H_0: \alpha_1 - \alpha_2 = 0 \Leftrightarrow$
$H_0: A \cdot \underset{\sim}{b} = \underset{\sim}{c}$

$A_{1\times 2} = (1, -1)$, $\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Ακολουθώ ίδια διαδικασία με προηγούμενη άσκηση.